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Graph equations for line graphs, middle graphs, total closed neighborhood graphs and total closed edge neighborhood graphs

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Abstract. Let G be a graph with vertex set $V(G)$, edge set $E(G)$. For each vertex (or edge) of G, a new vertex is taken and the resulting set of vertices is denoted by $V_1(G)$ (or $E_1(G)$) respectively. Let $\overline{G}$ and $L(G)$ denote the complement graph and line graph of G. The *middle graph* $M(G)$ as an intersection graph $\Omega(F)$ on the vertex set $V(G)$ of any graph G. Let $E(G)$ be the edge set of G and $F = V'(G) \cup E(G)$ where $V'(G)$ indicates the family of one-point subsets of the set $V(G)$, then $M(G) \cong \Omega(F)$.
The *total closed neighborhood graph* $N_{tc}(G)$ of a graph G is defined as the graph having vertex set $V(G) \cup V_1(G)$ and two vertices are adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u_i' of $V_1(G)$ and the other to a vertex w_j of G and w_j is in $N[u_i]$ (see [1]).
For a graph G, we define the *total closed edge neighborhood graph* $\mathrm{EN}_{tc}(G)$ of a graph G as the graph having vertex set $E(G) \cup E_1(G)$ with two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e_i' of $E_1(G)$ and the other to an element e_j of $E(G)$ where e_j is in $N[e_i]$.

In this paper, we solve the graph equations $L(G) \cong N_{tc}(H)$, $\overline{L(G)} \cong N_{tc}(H)$, $M(G) \cong N_{tc}(H)$, $\overline{M(G)} \cong N_{tc}(H)$, $L(G) \cong \mathrm{EN}_{tc}(H)$, $\overline{L(G)} \cong \mathrm{EN}_{tc}(H)$, $M(G) \cong \mathrm{EN}_{tc}(H)$ and $\overline{M(G)} \cong \mathrm{EN}_{tc}(H)$.
The symbol $\cong$ stands for isomorphism between two graphs.
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1 Introduction

By a graph, we mean a finite, undirected graph without loops or multiple edges. Definitions not given here may be found in [2]. For a graph G, let $V(G)$ and $E(G)$ denote its vertex set and edge set respectively.

Hamada and Yoshimura [3] defined a graph $M(G)$ as an intersection graph Ω(F) on the vertex set $V(G)$ of any graph G. Let $E(G)$ be the edge set of G and F = $V'(G) \cup E(G)$ where $V'(G)$ indicates the family of one-point subsets of the set $V(G)$. Let $M(G) \cong \Omega$(F). $M(G)$ is called the middle graph of G.

The *open-neighborhood* $N(u)$ of a *vertex* u in $V(G)$ is the set of all vertices adjacent to u.

$$N(u) = \{v/uv \in E(G)\}$$

The closed neighborhood $N[u]$ of a vertex u in $V(G)$ is given by

$$N[u] = \{u\} \cup N(u).$$

For each vertex u_i of G, a new vertex u_i' is taken and the resulting set of vertices is denoted by $V_1(G)$.

The *total closed neighborhood graph* $N_{tc}(G)$ of a graph G is defined as the graph having vertex set $V(G) \cup V_1(G)$ and two vertices are adjacent if they correspond to adjacent vertices of G or one corresponds to a vertex u_i' of $V_1(G)$ and the other to a vertex w_j of G and w_j is in $N[u_i]$ (see [1]).

The open-neighborhood $N(e_i)$ of an edge e_i in $E(G)$ is the set of edges adjacent to e_i.

$$N(e_i) = \{e_j/e_i \quad \text{and } e_j \text{ are adjacent in } G\}.$$

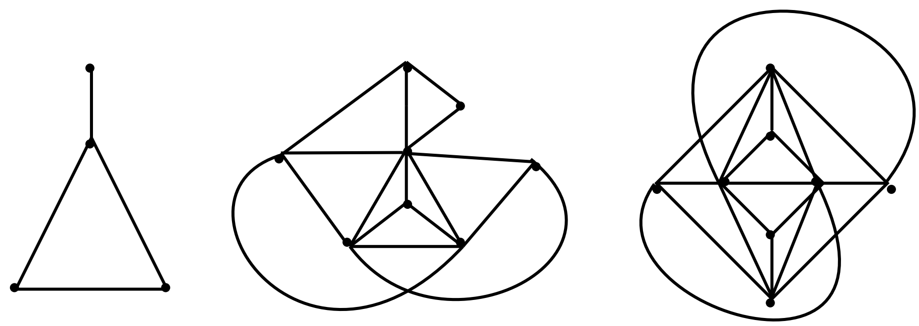
The closed-neighborhood $N[e_i]$ of an edge e_i in $E(G)$ is given by

$$N[e_i] = \{e_i\} \cup N(e_i)$$

For each edge e_i of G, a new vertex e'_i is taken and resulting set of vertices is denoted by $E_1(G)$.

For a graph G, we define the *total closed edge neighborhood graph* $\mathrm{EN}_{tc}(G)$ of a graph G as the graph having vertex set $E(G) \cup E_1(G)$ with two vertices are adjacent if they correspond to adjacent edges of G or one corresponds to an element e'_i of $E_1(G)$ and the other to an element e_j of $E(G)$, where e_j is in $N[e_i]$.

In Fig. 1, a graph G and its $N_{tc}(G)$ and $\mathrm{EN}_{tc}(G)$ are shown.



(a) (b) (c)

Figure 1: (a): G, (b): $N_{tc}(G)$ and (c) $\mathrm{EN}_{tc}(G)$.

The symbol $\cong$ stands for isomorphism between two graphs. Let $\overline{G}$, $L(G)$ and $T(G)$ denote respectively the complement, the line graph and the total graph of G. Cvetkoviè and Simiè [4] solved graph equations $L(G) \cong T(H), \overline{L(G)} \cong T(H)$. Akiyama et al. [5] solved graph equations $L(G) \cong M(H)$; $M(G) \cong T(H); \overline{M(G)} \cong T(H)$ and $\overline{L(G)} \cong M(H)$. Here we solve the following graph equations:

(1) $L(G) \cong N_{tc}(H)$.

(2) $\overline{L(G)} \cong N_{tc}(H)$.

(3) $M(G) \cong N_{tc}(H)$.

(4) $\overline{M(G)} \cong N_{tc}(H)$.

(5) $L(G) \cong \mathrm{EN}_{tc}(H)$.

(6) $\overline{L(G)} \cong \mathrm{EN}_{tc}(H)$.

(7) $M(G) \cong \mathrm{EN}_{tc}(H)$.

(8) $\overline{M(G)} \cong \mathrm{EN}_{tc}(H)$.

Beineke has shown in [6] that a graph G is a line graph if and only if G has none of the nine specified graphs F_i, $i = 1, 2, \ldots, 9$ as an induced subgraph. We depict here three of the nine graphs which are useful to extract our later results. These are $F_1 = K_{1,3}, F_2$ (see Fig. 2), and $F_3 = K_5 - x$, where x is any edge of K_5. A graph G^+ is the *endedge graph* of a graph G if G^+ is obtained from G by adjoining an endedge $u_i u_i'$ at each vertex u_i of G [5]. Hamada and Yoshimura [3] have proved that $M(G) \cong L(G^+)$.

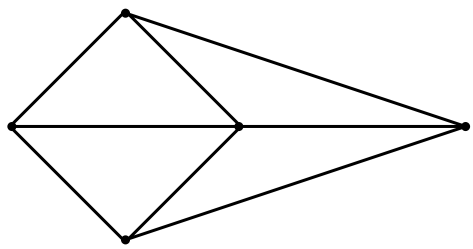


Figure 2: F_2.

2 The solution of $L(G) \cong N_{tc}(H)$

Any graph H which is a solution of the above equation, satisfies the following properties:

(i) H must be a line graph, since H is an induced subgraph of $N_{tc}(H)$.

(ii) H does not contain a cut-vertex, since otherwise, F_1 would be an induced subgraph of $N_{tc}(H)$.

(iii) H does not contain a component having more than two vertices, since otherwise, F_1 would be an induced subgraph of $N_{tc}(H)$.

It is not difficult to see from observation (ii) that H has no cut-vertices. We consider the following cases:
Case 1. Suppose H is connected. Then H is K_1 or K_2. The corresponding G is $K_{1,2}$ or $K_3 \circ K_2$ respectively.
Case 2. Suppose H is disconnected. Then H is nK_1 or nK_2. The corresponding G is $nK_{1,2}$ or $n(K_3 \circ K_2)$ respectively.

From the above discussion, we conclude the following

Theorem 2.1.

The following pairs (G,H) are all pairs of graphs satisfying the graph equation $L(G) = N_{tc}(H)$:

$$(nK_{1,2}, nK_1, \quad n \geq 1; \quad \text{and} \quad (n(K_3 \circ K_2), nK_2), \quad n \geq 1).$$

3 The solution of $\overline{L(G)} \cong N_{tc}(H)$

First, we observe that in this case H satisfies the following properties:

(i) If H has at least one edge, then it is connected, since otherwise, $\overline{F_1}$ and $\overline{F_2}$ are induced subgraphs of $N_{tc}(H)$.

(ii) H does not contain a path P_4 as an induced subgraph, since otherwise, $\overline{F_1}$ is an induced subgraph of $N_{tc}(H)$.

(iii) H does not contain C_n, $n \geq 5$ as an induced subgraph, since otherwise, $\overline{F_1}$ would be an induced subgraph of $N_{tc}(H)$.

(iv) H does not contain more than one cut-vertex, since otherwise, $\overline{F_1}$ would be an induced subgraph of $N_{tc}(H)$.

(v) H does not contain $K_{1,4}$ as an induced subgraph, since otherwise, $\overline{F_3}$ would be an induced subgraph of $N_{tc}(H)$.

(vi) H does not contain a cut-vertex which lies on blocks other than K_2, since otherwise, $\overline{F_2}$ is an induced subgraph of $N_{tc}(H)$.

Thus H has at most one cut-vertex. We consider the following cases:

Case 1. Suppose H has exactly one cut-vertex. Then H is $K_{1,2}$ or $K_{1,3}$. Corresponding G is $(C_4 \circ K_2) \cup K_2$ or $(K_4 \circ K_2) \cup K_2$ respectively.

Case 2. Suppose H has no cut-vertices. We consider the following subcases:

Subcase 2.1. $H = K_n$. In this case $(K_{1,n} \cup nK_2, K_n)$, $n \geq 1$ and $(K_3 \cup 3K_2, K_3)$ are the solutions.

Subcase 2.2. $H = K_{m,n}$. Then from observation (v), $(C_4 \circ K_4, K_{2,3})$ and $(K_4 \circ K_4, K_{3,3})$ are the solutions.

Subcase 2.3. H is neither a complete graph nor a complete bipartite graph. From observation (iii), H is C_n, $n \leq 4$ or $K_4 - x$, where x is any edge of K_4. In this case the solutions are $(K_{1,3} \cup 3K_2, C_3)$, $(K_3 \cup 3K_2, C_3)$, $(C_4 \circ C_4, C_4)$ and $(G', K_4 - x)$ where G' is the graph shown in Fig. 3 are the solutions.

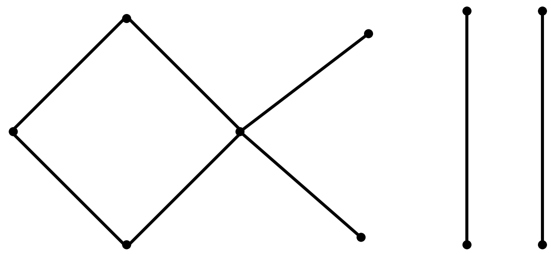


Figure 3: G'.

Thus we have the following

Theorem 3.1. *The following pairs (G,H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong N_{tc}(H)$:*

$((C_4 \circ K_2) \cup K_2, K_{1,2})$; $((K_4 \circ K_2) \cup K_2, K_{1,3})$; $(K_{1,n} \cup nK_2, K_n)$, $n \geq 1$; $(K_3 \cup 3K_2, K_3)$; $(C_4 \circ K_4, K_{2,3})$; $(K_4 \circ K_4, K_{3,3})$; $(C_4 \circ C_4, C_4)$; and $(G', K_4 - x)$, where x is any edge of K_4 and G' is the graph shown in Fig. 3.

4 The solution of $M(G) \cong N_{tc}(H)$

Theorem 2.1 gives solutions of the graph equation $L(G) \cong N_{tc}(H)$. But none of these is of the form (G^+, H). Hence, there is no solution of the equation $M(G) \cong N_{tc}(H)$. Now, we state the following result.

Theorem 4.1. *There is no solution of the graph equation $M(G) \cong N_{tc}(H)$.*

5 The solution of $\overline{M(G)} \cong N_{tc}(H)$

Theorem 3.1 gives solution of the equation $\overline{L(G)} \cong N_{tc}(H)$. But none of these is of the form (G^+, H). Therefore there is no solution of the graph equation $\overline{M(G)} \cong N_{tc}(H)$. Now, we state the following result.

Theorem 5.1. *There is no solution of the graph equation $\overline{M(G)} \cong N_{tc}(H)$.*

6 The solution of $L(G) \cong \mathrm{EN}_{tc}(H)$

In this case, H satisfies the following properties:

(i) H does not contain a cycle C_n, $n \geq 3$ as a subgraph, since otherwise, F_1 is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(ii) H does not contain a component having more than one cut-vertex, since otherwise, F_1 is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(iii) The maximum degree of H does not exceed two, since otherwise, F_1 is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(iv) H does not contain a cut-vertex which lies on more than two blocks, since otherwise, F_1 is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(v) H does not contain a cut-vertex which lies on a block other than K_2, since otherwise, F_1 is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

From observation (ii), it follows that every component of H has at most one cut-vertex. We consider the following cases:

Case 1. Suppose H has no cut-vertices. Then from observation (i), H is nK_2, $n \geq 1$. The corresponding G is $nK_{1,2}$, $n \geq 1$.

Case 2. Suppose H has cut-vertex. We consider the following subcases:

Subcase 2.1. Assume H is connected. Then H is $K_{1,2}$. The corresponding G is $K_3 \circ K_2$.

Subcase 2.2. Assume H is disconnected. Then H is $nK_{1,2} \cup mK_2$, $m \geq 0$, $n \geq 1$. The corresponding G is $n(K_3 \circ K_2) \cup mK_{1,2}$. From above discussions, we conclude the following:

Theorem 6.1. *The following pairs (G,H) are all pairs of graphs satisfying the graph equation $L(G) \cong \mathrm{EN}_{tc}(H)$:*

$$(nK_{1,2}, nK_2), \quad n \geq 1; \quad (K_3 \circ K_2, K_{1,2}); \quad \textit{and}$$
$$(n(K_3 \circ K_2) \cup mK_{1,2}, \quad nK_{1,2} \cup mK_2), \quad m \geq 0, \quad n \geq 1.$$

7 The solution of $\overline{L(G)} \cong \mathrm{EN}_{tc}(H)$

In this case, H satisfies the following properties:

(i) If H is disconnected, then it has at most three components, each of which is K_2 since otherwise, $\overline{F_3}$ is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(ii) H is not a path P_n, $n \geq 5$ since otherwise, $\overline{F_1}$ is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(iii) H does not contain C_n, $n \geq 5$, since otherwise, $\overline{F_2}$ is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(iv) H is not a complete bipartite graph $K_{m,n}$, for $m \geq 3$ or $n \geq 3$, since otherwise, $\overline{F_2}$ is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

(v) H does not contain more than two cut-vertices, since otherwise, $\overline{F_1}$ is an induced subgraph of $\mathrm{EN}_{tc}(H)$.

Thus H has at most two cut-vertices. We consider the following cases:
Case 1. If H has exactly one cut-vertex, then H is $K_{1,n}$, $n \geq 1$ or $K_3 \circ K_2$.
For $H = K_{1,n}$, $n \geq 1$, $G = K_{1,n} \cup nK_2$
For $H = K_3 \circ K_2$, G is a graph as shown in Fig. 3.
Case 2. If H has exactly two cut-vertices. Then H is a path P_4. Corresponding G is $(C_4 \circ K_2) \cup K_2$.
Case 3. If H has no cut-vertices. We consider the following subcases:
Subcase 3.1. If H is disconnected. Then from observation (i), H is nK_2 , $n \leq 3$. For $n = 1$, $H = K_2$ and $G = 2K_2$. For $n = 2$, $H = 2K_2$ and $G = C_4$. For $n = 3$, $H = 3K_2$ and $G = K_4$.
Subcase 3.2. If H is connected. We consider the following subcases.
Subcase 3.2.1. $H = K_n$. In this case, it follows from observation (iii), that $(2K_2, K_2), (K_3 \cup 3K_2, K_3), (K_{1,3} \cup 3K_2, K_3)$ and (G', K_4) where G' is the graph shown in Fig. 4 are the solutions.

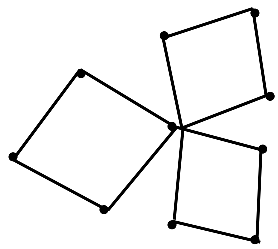


Figure 4:

Subcase 3.2.2. $H = K_{m,n}$. Then from observation (iv), $(2K_2, K_{1,1}), (K_{1,2} \cup 2K_2, K_{1,2})$ and $(C_4 \circ C_4, K_{2,2})$ are the solutions.
Subcases 3.2.3. H is neither a complete graph nor a complete bipartite graph. From observation (iii), H is C_n, $n \leq 4$ or $K_4 - x$, where x is any edge of K_4. In this case

$(K_3 \cup 3K_2, C_3), (K_{1,3} \cup 3K_2, C_3), (C_4 \circ C_4, C_4)$ and $(G', K_4 - x)$, where G' is the graph as shown in Fig. 5, are the solutions.

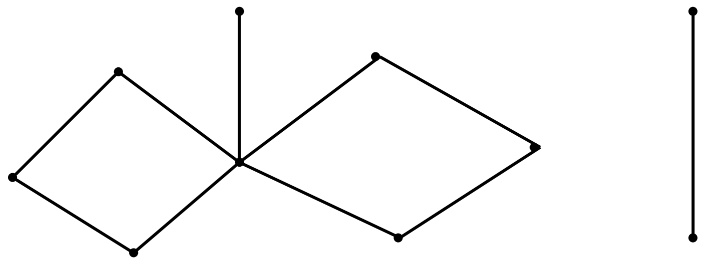


Figure 5:

Thus the graph equation is solved and we have the following

Theorem 7.1. *The following pairs (G,H) are all pairs of graphs satisfying the graph equation $\overline{L(G)} \cong \mathrm{EN}_{tc}(H)$:*
$(K_{1,n} \cup nK_2, K_{1,n})$, $n \geq 1$; $((C_4 \circ K_2) \cup K_2, P_4)$; $(C_4, 2K_2)$; $(K_4, 3K_2)$;
$(K_3 \cup 3K_2, K_3)$; $(K_{1,3} \cup 3K_2, K_3)$; $(C_4 \circ C_4, C_4)$; $(G', K_3 \circ K_2)$,
where G' is the graph as shown in Fig. 3; (G', K_4), where G' is the graph as shown in Fig. 4; and $(G', K_4 - x)$, where G' is the graph as shown in Fig. 5.

8 The solution of $M(G) \cong \mathrm{EN}_{tc}(H)$

Theorem 6.1 gives solutions of the equation $L(G) \cong \mathrm{EN}_{tc}(H)$. But none of these is of the form (G^+, H). Hence there is no solution of the equation $M(G) \cong \mathrm{EN}_{tc}(H)$. Thus we obtain the following result.

Theorem 8.1. *There is no solution of the graph equation $M(G) \cong \mathrm{EN}_{tc}(H)$.*

Theorem 7.1 gives the solution of the graph equation $\overline{L(G)} \cong \mathrm{EN}_{tc}(H)$. Among these only one solution $(2K_2, K_2)$ is of the form (G^+, H). Therefore, the solution of the equation $\overline{M(G)} \cong \mathrm{EN}_{tc}(H)$ is $(2K_1, K_2)$. Thus we have the following result.

Theorem 8.2. *There is only one solution* $(2K_1, K_2)$ *of the graph equation* $\overline{M(G)} \cong$ $\mathrm{EN}_{tc}(H)$.

References

[1] V. R. Kulli and Nanda S. Warad, *On the total closed neighborhood graph of a graph*, J. of Discrete Math. Sci. and Cryptography, **4** (2001), No. 2–3, 109–144.

[2] F. Harary, *Graph Theory*, Addison-Wesley, Reading, Mass, (1969).

[3] T Hamada and I. Yoshimura, *Traversability and connectivity of the middle graph of a graph*, Discrete Math. **14** (1976) 247–256.

[4] D. M. Cvetkoviè and S. K. Simiè, *Graph equations for line graphs and total graphs*, Discrete Math. **13** (1975) 315–320.

[5] T. Akiyama, T. Hamada and I. Yoshimura, *Graph equations for line graphs, total graphs and middle graphs* TRU Math. **12** (2) (1976).

[6] L. W. Beineke, *Derived graphs and digraphs*, in : H. Sachs, V. Voss and H. Walther, eds. Beiträge Zur Graphentheorie (Teubner, Leipzig. 1968) 17–33.